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IMPROVED INEQUALITIES FOR BALANCED INCOMPLETE BLOCK DESIGNS

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It is known that there are some lower bounds for the number of blocks in a balanced incomplete block design (BIBD). Especially, Fisher's inequality $b \geqslant v$ is well-known for a BIBD with parameters v, b, r, k and λ. Fisher's inequality can be improved upon if one puts additional restrictions on a BIBD. Artifical restrictions are infinite in number so is the number of new bounds. The condition of non-symmetry on the design discussed here is a very simple restriction. The main purpose of this paper is to give improvements of inequalities for BIBDs with the only condition of non-symmetry. Improved inequalities appear to be the best for any non-symmetrical BIBD.

1. Introduction and summary

Relations of various inequalities among parameters of a BIBD, an α-resolvable BIBD and a BIBD with some identical blocks have been derived by Bose [1, 2], Fisher [3], Kageyama [4–6], Khatri and Shah [7], Kishen and Rao [8], Mann [9], Mikhail [10], Murty [11], Nair [12], Roy [13], Stanton [14], Stanton and Sprott [15] and others. However, little attention has been given to discussions of an inequality for the condition under which a BIBD is non-symmetrical. The reason for this is probably that the only condition of non-symmetry may not characterize the structure of a BIBD (like the block structure). The condition of non-symmetry on a design is however a very simple restriction and discussions under the condition should be remarked from a point of view of an improvement of an inequality.

In this paper, various inequalities for the number of blocks in non-symmetrical BIBDs are given and further compared in an arithmetical sense of Kishen and Rao [8]. Improved inequalities appear to be the best for any non-symmetrical BIBD.

2. Improvements

For a BIBD with parameters v, b, r, k and λ in which $vr = bk$, $\lambda(v-1) = r(k-1)$, it is known that the following inequalities hold:

$$b \geq v \qquad \text{(cf. [5])}.$$

$$b \geq v + r - k \qquad \text{(cf. [8])}.$$

$$b \geq 1 + \frac{k(r-1)^2}{r-k+\lambda(k-1)} \; (=1+A, \text{ say}) \qquad \text{(cf. [12])}. \tag{1}$$

$$b \geq 1 + \frac{(v-k)(b-r-1)^2}{b-v-r+k+(b-2r+\lambda)(v-k-1)} \; (=1+B, \text{ say}) \qquad \text{(cf. [8])}. \tag{2}$$

Furthermore, it is clear that each equality sign in these four inequalities holds when and only when the BIBD is symmetrical (i.e., $b=v$). Hence, for a non-symmetrical (i.e., $b>v$) BIBD, we can easily obtain the following modified inequalities:

$$b \geq 2 + \left[\frac{k(r-1)^2}{r-k+\lambda(k-1)}\right], \tag{3}$$

$$b \geq 2 + \left[\frac{(v-k)(b-r-1)^2}{b-v-r+k+(b-2r+\lambda)(v-k-1)}\right], \tag{4}$$

where symbol $[x]$ denotes the greatest integer $\leq x$ throughout this paper. It is shown by Kishen and Rao [8] that if $v \geq 2k$, then $b \geq 1+(v-k)(b-r-1)^2/\{b-v-r+k+(b-2r+\lambda)(v-k-1)\} \geq 1+k(r-1)^2/\{r-k+\lambda(k-1)\}$. Hence, if $v \geq 2k$, then (4) is more stringent than (3). Thus, inequality (4) may be reasonable to be used when $v \geq 2k$. Incidentally, note that $b \geq 1+k(r-1)^2/\{r-k+\lambda(k-1)\} \geq v+r-k \geq v$ for any BIBD.

Remark 2.1. The idea of this derivation can be generalized as follows: Since $b>1+A$ and $b>1+B$ for a non-symmetrical BIBD, we have $pb>p+pA$ and $pb>p+pB$ for any positive integer p. Furthermore, since pA and pB are not necessarily integers, we obtain $pb \geq p+1+[pA]$ and $pb \geq p+1+[pB]$. Hence we can have $b \geq 1+1/p+[pA]/p$ and $b \geq 1+1/p+[pB]/p$ for any positive integer p. Note that when $p=1$, these inequalities imply (3) and (4).

As an improvement of an inequality for a BIBD with a simple restriction we first have

Lemma 2.2. *For a non-symmetrical* BIBD *with parameters v, b, r, k and λ, when $v=nk$ for some integer $n \geq 2$, $b \geq n^2\lambda+n$ $(\geq v+r-1)$ holds, and when $v=mk+l$ for some integer $m \geq 1$ and $0<l<k$,*

$$b \geq \max\{v+v/g, (m+1)r-m\lambda\}$$

holds, where $g=(v,k)<k$.

Proof. The inequality $b \geq n^2\lambda + n(\geq v + r - 1)$ is already given and discussed with examples (cf. [1, 6]). Firstly, the $(v, k) = g$ leads to an expression $v = v_1 g$, $k = k_1 g$, $(v_1, k_1) = 1$. Relation $vr = bk$ implies that k_1 must divide r, i.e., v_1 must divide b. Now, let $b = pv_1$. Then $pv_1 = b > v = gv_1$ yields $p > g$, i.e., $p \geq g + 1$. Hence $b \geq (g+1)v_1 = v + v/g$. If $g = k$, then there exists a positive integer n (≥ 2) such that $v = nk$. Hence $b \geq v + n$. However, since $vr = bk$, $\lambda(v-1) = r(k-1)$ and $v = nk$ yield $r - n\lambda \geq 1$, we can show that $v + r - 1 \geq v + n$. Thus, when $v = nk$ for some integer $n \geq 2$, the first inequality is more suitable. Secondly, from $vr = bk$, $\lambda(v-1) = r(k-1)$ and $v = mk + l$ we have $b - r = (r-\lambda)(v-1)/k = m(r-\lambda) + (l-1)(r-\lambda)/k$, which leads from $l \geq 1$ to $b \geq (m+1)r - m\lambda$ for $m \geq 1$. This bound is always attained by a BIBD with $v = mk + 1$.

Example 2.3. Consider a BIBD with $v = 16$, $b = 24$, $r = 9$, $k = 6$ and $\lambda = 3$ (see [16]). Then $b \geq v + v/g$ and $b \geq (m+1)r - m\lambda$ imply $24 \geq 24$ and $24 \geq 21$, respectively.

Example 2.4. Consider a BIBD with $v = 10$, $b = 30$, $r = 9$, $k = 3$ and $\lambda = 2$ (see [16]). Then $b \geq v + v/g$ and $b \geq (m+1)r - m\lambda$ imply $30 \geq 20$ and $30 \geq 30$, respectively.

Example 2.5. Consider a BIBD with $v = 16$, $b = 40$, $r = 15$, $k = 6$ and $\lambda = 5$ (see [16]). Then $b \geq v + v/g$ and $b \geq (m+1)r - m\lambda$ imply $40 \geq 24$ and $40 \geq 35$, respectively.

The last example shows that much yet remains to be improved. It is interesting to note that $b \geq v + v/g$ in Lemma 2.2 can also be improved to $b \geq v + v/g + r - k - k/g$, by using $r \geq k + k/g$ (which is equivalent to $b \geq v + v/g$) for the parameters of the complementary BIBD, since $g = (v, k) = (v, v-k)$. This approach is essentially the same as a method [8] of deriving $b \geq v + r - k$ from $b \geq v$ for a BIBD.

We next deal with comparisons among the inequality of Lemma 2.2, inequalities (3) and (4).

Lemma 2.6. *For a non-symmetrical* BIBD *with parameters v, b, r, k and λ, if $v < 2k$, then*

$$b \geq 2 + \left[\frac{k(r-1)^2}{r - k + \lambda(k-1)}\right] \geq v + \frac{v}{g}$$

holds, where $g = (v, k)$.

Proof. Let $F = 2 + k(r-1)^2/\{r - k + \lambda(k-1)\} - v - v/g$. Then it is sufficient to show that $F \geq 0$, and that $F = 0$ holds only if $k(r-1)^2/\{r - k + \lambda(k-1)\}$ is an integer. Now we can write assumption $v < 2k$ as $v = 2k - l$ for a positive integer l, which from $vr=bk$ implies $b=2r-lr/k$. Thus, there exists an integer α such that $lr=\alpha k$, i.e., $r=\alpha k/l$. Furthermore, $b \geqslant v+v/g$ and $vr=bk$ lead to $r \geqslant k+k/g=(1+1/g)k$. Hence we get from $r=\alpha k/l$

$$\frac{\alpha}{l} \geqslant 1+\frac{1}{g}. \tag{5}$$

Then from $vr=bk$, $\lambda(v-1)=r(k-1)$, $v=2k-l$ and $r=\alpha k/l$ we have

$$F=2+\frac{k(r-1)^2}{r-k+\lambda(k-1)}-v-\frac{v}{g}=\frac{F'}{g\{r-k+\lambda(k-1)\}},$$

where $F'=(g-1)\lambda(k-1)+k\{(\alpha^2 g/l^2-\alpha(g+1)/l)k^2+2(1+g-\alpha g/l)k+(\alpha-l)\times(g+1)+\alpha(g-1)/l-g\}$. Furthermore, since α and l are positive integers, relation (5) can be written as $\alpha=(1+1/g)l+x$ for a non-negative real number x, and so we obtain $F'=(g-1)\lambda(k-1)+k\{xk(gk-2g+k+xgk/l)/l+l+x(g+1)+(l-1)/g+x(g-1)/l\}$ which is clear to be positive from $k \geqslant 2$, $g \geqslant 1$ and $l \geqslant 1$, i.e., $F'>0$. Hence $F>0$. That is, $2+[k(r-1)^2/\{r-k+\lambda(k-1)\}]-v-v/g$ is non-negative or positive accordingly as the expression $F'-g\{r-k+\lambda(k-1)\}$ is negative or non-negative. This completes the proof.

Noting that $v<2k$, $v=mk+l$ and $1 \leqslant l \leqslant k-1$ imply $m=1$, as a comparison between $b \geqslant (m+1)r-m\lambda$ and (3) when $v<2k$ we have from Lemmas 2.2 and 2.6

Theorem 2.7. *For a non-symmetrical* BIBD *with parameters v, b, r, k and λ, if $v<2k$, then*

$$b \geqslant \max\left\{2+\left[\frac{k(r-1)^2}{r-k+\lambda(k-1)}\right], 2r-\lambda\right\}$$

holds.

Example 2.8. Consider a BIBD with $v=3$, $b=12$, $r=8$, $k=2$ and $\lambda=4$. Then $b \geqslant 2+[k(r-1)^2/\{r-k+\lambda(k-1)\}]$ and $b \geqslant 2r-\lambda$ imply $12 \geqslant 11$ and $12 \geqslant 12$, respectively.

Note that the bound of $b \geqslant 2+[k(r-1)^2/\{r-k+\lambda(k-1)\}]$ is attained in almost all BIBDs with $v<2k$ except possibly for trivial BIBDs like Example 2.8. We believe the inequality in Theorem 2.7 to be the best possible for any non-symmetrical BIBD with $v<2k$.

Lemma 2.9. *For a non-symmetrical* BIBD *with parameters v, b, r, k and λ, if $v=nk$ for an integer $n \geqslant 2$,*

$$b \geqslant 2+\left[\frac{(v-k)(b-r-1)^2}{b-v-r+k+(b-2r+\lambda)(v-k-1)}\right] \geqslant n^2\lambda+n$$

holds.

Proof. Let $F=2+(v-k)(b-r-1)^2/\{b-v-r+k+(b-2r+\lambda)(v-k-1)\}-n^2\lambda-n$. Then it is sufficient to show that $F\geqslant 0$, and that $F=0$ holds only if $(v-k)(b-r-1)^2/\{b-v-r+k+(b-2r+\lambda)(v-k-1)\}$ is an integer. When $v=nk$ for $n\geqslant 2$, it is known (cf. [6]) that the following relations hold:

$$r=n\lambda+p \quad \text{and} \quad n\lambda=\lambda+pk-p \tag{6}$$

for a positive integer p. Now we have from $b=nr$

$$\begin{aligned}F&=2+\frac{(v-k)(b-r-1)^2}{b-v-r+k+(b-2r+\lambda)(v-k-1)}-\frac{b^2\lambda}{r^2}-\frac{b}{r}\\&=\frac{F'}{r^2\{b-v-r+k+(b-2r+\lambda)(v-k-1)\}},\end{aligned}$$

where

$$F'=(2r^2-b^2\lambda-br)\{b-v-r+k+(b-2r+\lambda)(v-k-1)\}+r^2(v-k)(b-r-1)^2.$$

By use of $v=nk$, $b=nr$ and (6), we after some calculations have

$$\begin{aligned}F'/r^2={}&-(3n-4)pk+(n-1)^2k+n(n^2-n-1)pk\lambda\\&-(n-1)(n-2)k\lambda+(n+1)(n^2-3n+1)p^2k^2-n(n-1)(n-2)pk^2\\&+(n+1)p^2k+(n-1)^2(n-2)k\lambda^2+(n-1)(n^2-5n+5)pk^2\lambda\\&-(n-1)(n-3)p^2k^3,\end{aligned}$$

and from $\lambda=p(k-1)/(n-1)$ in (6) we obtain

$$\begin{aligned}F'/r^2k&=-2(n-1)p+(n-1)^2-(n-2)(n^2-n+1)pk+n(n-2)(n^2-n+1)\\&\quad\times p^2k/(n-1)-(n^3-3n^2+2n-1)p^2/(n-1)\\&=p(n-2)(n^2-n+1)\{n(p-1)+1\}k/(n-1)+\{(n-1)^3-2(n-1)^2\\&\quad\times p-(n^3-3n^2+2n-1)p^2\}/(n-1).\end{aligned}\tag{7}$$

Since $k\geqslant 2$ and $p(n-2)(n^2-n+1)\{n(p-1)+1\}/(n-1)\geqslant 0$, we also get

$$\begin{aligned}F'/r^2k\geqslant{}&\{2p(p-1)n^4-(7p^2-8p-1)n^3+(6p^2-11p-3)n^2\\&-(6p^2-14p-3)n+p^2-6p-1\}/(n-1).\end{aligned}\tag{8}$$

(i) When $p=1$, from (8) $F'/r^2k\geqslant(n-2)(2n^2-4n+3)/(n-1)\geqslant 0$ since $n\geqslant 2$. Hence in this case $F'\geqslant 0$ and so $F\geqslant 0$. Note that $F=0$ holds if and only if $p=1$ and $n=2$ which implies that $(v-k)(b-r-1)^2/\{b-v-r+k+(b-2r+\lambda)\times(v-k-1)\}$ is integral. (ii) When $n=2$, from (7) $F'/r^2k=(p-1)^2\geqslant 0$ since $p\geqslant 1$. Hence in this case $F'\geqslant 0$ and so $F\geqslant 0$. Also note that $F=0$ holds if and only if $p=1$ and $n=2$. (iii) When $p=2$, from (8) $F'/r^2k\geqslant\{4(n-3)n^3+n(n^2-n+7)-9\}/(n-1)\geqslant 30/(n-1)>0$ since $n\geqslant 3$ by (ii). Hence in this case $F'>0$ and so $F>0$. (iv) When $n=3$, from (8) $F'/r^2k\geqslant\frac{1}{2}(10p^2-9p+8)>0$ since $p\geqslant 3$ by (i) and (iii). Hence in this case $F'>0$ and so $F>0$. (v) The remaining case is one where $n\geq 4$ and $p\geq 3$. From (8)

$$F'/r^2k \geq \{n^3[(2n-7)p(p-1)+p+1]+n(n-1)[6p(p-2)+p-3] \\ +p^2+(3n-6)p-1\}/(n-1)$$

which can be shown to be positive from $n\geq 4$ and $p\geq 3$. Hence in this case $F'>0$ and so $F>0$. This completes the proof.

Lemma 2.10. *For a non-symmetrical* BIBD *with parameters* v, b, r, k *and* λ, *if* $v\geq 2k$, *then*

$$b\geq 2+\left[\frac{(v-k)(b-r-1)^2}{b-v-r+k+(b-2r+\lambda)(v-k-1)}\right]\geq v+\frac{v}{g} \tag{9}$$

holds, where $g=(v,k)$.

The proof of this lemma follows from some tedious calculations similar to that of Lemma 2.6. We, however, give another simple proof of this lemma as follows. It is clear that the complement of a BIBD with parameters v, b, r, k, λ and with $v\geq 2k$ is a BIBD with parameters $v^*=v$, $b^*=b$, $r^*=b-r$, $k^*=v-k$, $\lambda^*=b-2r+\lambda$ and with $v^*\leq 2k^*$. In this case, from $g=(v,k)=(v,v-k)=(v^*,k^*)$ relation (9) becomes that

$$b^*\geq 2+\left[\frac{k^*(r^*-1)^2}{r^*-k^*+\lambda^*(k^*-1)}\right]\geq v^*+\frac{v^*}{g} \tag{10}$$

for a non-symmetrical BIBD with $v^*\leq 2k^*$. Hence Lemma 2.6 shows that (10) is valid for $v^*<2k^*$, i.e., relation (9) holds for $v>2k$. It is therefore sufficient to show that relation (9) holds for $v=2k$. When $v=2k$, we have $g=k$ and $b=2r$. Hence (9) becomes

$$b\geq 2+\left[\frac{k(r-1)^2}{r-k+\lambda(k-1)}\right]\geq 2k+2. \tag{11}$$

Furthermore, Lemma 2.9 shows that $b\geq 2+[k(r-1)^2/\{r-k+\lambda(k-1)\}]\geq 4\lambda+2$. Now from (6) and $k\geq 2$ we can show that $4\lambda+2-(2k+2)\geq 4(p-1)\geq 0$ since $p\geq 1$ in (6). This implies that relation (11) holds. Thus, the proof of Lemma 2.10 is completed.

Therefore, as a comparison between $b\geq(m+1)r-m\lambda$ and (4) when $v\geq 2k$, we have from Lemmas 2.2, 2.9 and 2.10:

Theorem 2.11. *For a non-symmetrical* BIBD *with parameters* v, b, r, k *and* λ, *if* $v\geq 2k$, *then*

$$b\geq\max\left\{2+\left[\frac{(v-k)(b-r-1)^2}{b-v-r+k+(b-2r+\lambda)(v-k-1)}\right],(m+1)r-m\lambda\right\},$$

where $v=mk+l$ *and* $1\leq l\leq k-1$.

Example 2.12. Consider a BIBD with $v=5$, $b=20$, $r=8$, $k=2$ and $\lambda=2$. Then $b \geqslant 2+[(v-k)(b-r-1)^2/\{b-v-r+k+(b-2r+\lambda)(v-k-1)\}]$ and $b \geqslant (m+1)r-m\lambda$ imply $20 \geqslant 19$ and $20 \geqslant 20$, respectively.

Note that the bound of $b \geqslant 2+[(v-k)(b-r-1)^2/\{b-v-r+k+(b-2r+\lambda)\times(v-k-1)\}]$ is attained in almost all BIBDs with $v \geqslant 2k$ except possibly for trivial BIBDs like Example 2.12 and BIBDs with $v=mk+1$ formed by taking some copies of a BIBD like Examples 4.1 and 4.2. We believe the inequality in Theorem 2.11 to be the best possible for any non-symmetrical BIBD with $v \geqslant 2k$.

3. Application

A BIBD with parameters v, $b=\beta t$, $r=\alpha t$, k and λ is called α-resolvable if the blocks can be separated into t sets of β blocks each such that each set contains every treatment exactly α times (cf. [5]). A 1-resolvable design is simply called resolvable (cf. [1]). Moreover, if two blocks belonging to different sets have the same number of treatments in common, the resolvable design is called affine resolvable. In this case we have from Theorem 2.11:

Corollary 3.1. *For a resolvable* BIBD *with parameters $v=nk$, b, r, k, λ and with an integer $n \geqslant 2$, an inequality*

$$b \geqslant 2+\left[\frac{(v-k)(b-r-1)^2}{b-v-r+k+(b-2r+\lambda)(v-k-1)}\right]$$

holds.

On the other hand, for a resolvable BIBD with parameters $v=nk$, b, r, k, λ and a positive integer $n \geqslant 2$, the following inequalities are known: $b \geqslant v+r-1$ (cf. [1], [13]), $b \geqslant n^2\lambda+n$ (cf. [6]) and $b \geqslant rk(r-1)/\{r-k+\lambda(k-1)\}$ (cf. [12]). As a comparison of these bounds, we have

Lemma 3.2. *For a resolvable* BIBD *with parameters $v=nk$, b, r, k and λ, relations*

$$b \geqslant \frac{rk(r-1)}{r-k+\lambda(k-1)} \geqslant 4\lambda+2 \geqslant v+r-1, \quad \text{for } n=2,$$

$$b \geqslant n^2\lambda+n \geqslant \frac{rk(r-1)}{r-k+\lambda(k-1)} \geqslant v+r-1, \quad \text{for } n \geqslant 3$$

hold. The equality signs hold at the same time when and only when the BIBD *is affine resolvable in each case.*

Proof. Since $r>k$ for a resolvable BIBD and

$$n^2\lambda+n-\frac{rk(r-1)}{r-k+\lambda(k-1)}\ (=F,\text{ say})$$
$$=\frac{\lambda\{r-k+\lambda(k-1)\}n^2+\{r-k+\lambda(k-1)\}n-rk(r-1)}{r-k+\lambda(k-1)},$$

let $\{r-k+\lambda(k-1)\}F=G$. Then $G=\lambda\{r-k+\lambda(k-1)\}n^2+\{r-k+\lambda(k-1)\}n-rk(r-1)$. Relations $vr=bk$, $\lambda(v-1)=r(k-1)$ and $v=nk$ yield $(r-n\lambda)k=r-\lambda>0$. Hence $r-n\lambda\geq 1$. Now, if $r-n\lambda=1$, then from $vr=bk$, $\lambda(v-1)=r(k-1)$ and $v=nk$ we obtain $n\lambda=\lambda+k-1$, and then $r=\lambda+k$. These relations lead to $G=0$, i.e., $F=0$ provided $r-n\lambda=1$. In general, if $r-n\lambda=p$ $(p\geq 1)$, then we have $n\lambda=\lambda+p(k-1)$, and then $r=\lambda+pk$. In this case, these relations on slight readjustment, yield $G=k(p-1)\{(n-2)n\lambda+n-p\}$. Since $k\geq 2$, from $n\lambda=\lambda+(k-1)p$ we have $n\lambda>p\geq 1$. Hence, if $n\geq 3$, then $G\geq 0$, i.e., $F\geq 0$ for any $p\geq 1$ (the second relation in the lemma). If $n=2$, then $G=-k(p-1)(p-2)\leq 0$ for any $p\geq 1$, i.e., $F\leq 0$ (the first relation in the lemma). Furthermore, since $rk(r-1)/\{r-k+\lambda(k-1)\}\geq v+r-1$ and $n^2\lambda+n\geq v+r-1$ are shown in [12] and [6], respectively, the relations of the lemma follow. It is clear that the equality signs hold at the same time when and only when the BIBD is affine resolvable.

As a comparison of inequalities described here, we have

Statement 3.3. *The inequality of Corollary* 3.1 *appears to be the best for any resolvable* BIBD.

Explanation. From Lemmas 2.9 and 3.2 it is clear that the required inequality appears to be the best for $n\geq 3$. It is therefore sufficient to show from Lemma 3.2 that when $n=2$, relation

$$b\geq 2+[(v-k)(b-r-1)^2/\{b-v-r+k+(b-2r+\lambda)(v-k-1)\}]$$
$$\geq rk(r-1)/\{r-k+\lambda(k-1)\}$$

holds. Let

$$F=2+(v-k)(b-r-1)^2/\{b-v-r+k+(b-2r+\lambda)(v-k-1)\}$$
$$-rk(r-1)/\{r-k+\lambda(k-1)\}.$$

From $v=2k$, $b=2r$ and (6) we have

$$F=2+\frac{k(r-1)^2}{r-k+\lambda(k-1)}-\frac{rk(r-1)}{r-k+\lambda(k-1)}=\frac{k(p-1)}{r-k+\lambda(k-1)}.$$

If $p>1$, then $F>0$ and so the required relation clearly holds. If $p=1$, then $F=0$. In this case, since both $k(r-1)^2/\{r-k+\lambda(k-1)\}$ and $rk(r-1)/\{r-k+\lambda(k-1)\}$

are integers, the required relation holds. Thus, the inequality of Corollary 3.1 is the best possible for any resolvable BIBD as compared with all the inequalities which we can know as yet.

Note that as seen from Lemma 2.9, the bound of the inequality of Corollary 3.1 is of course attained for any affine resolvable BIBD, and further attained for some resolvable BIBDs which are not affine resolvable. This is a remarkable property of the inequality of Corollary 3.1.

In general, it is known (cf. [6]) that for an α-resolvable BIBD with parameters v, $b=\beta t$, $r=\alpha t$, k and λ, an inequality $b \geq \max\{v+t-1, (\beta^2\lambda+\beta)/\alpha^2\}$ holds and the inequality appears to be the best for any α-resolvable BIBD with $\alpha \geq 2$. As a further improvement of the inequality, we have from Theorem 2.11:

Corollary 3.4. *For an α-resolvable* BIBD *with parameters v, $b=\beta t$, $r=\alpha t$, k and λ, an inequality*

$$b \geq 2+\left[\frac{(v-k)(b-r-1)^2}{b-v-r+k+(b-2r+\lambda)(v-k-1)}\right]$$

holds.

Statement 3.5. *The inequality of Corollary* 3.4 *appears to be the best for any α-resolvable* BIBD *with parameters v, $b=\beta t$, $r=\alpha t$, k and λ, provided β is divisible by α.*

Explanation. When α divides β, since $v\alpha=\beta k$, there exists an integer n $(=\beta/\alpha)$ such that $v=nk$. Hence, it follows from Lemmas 2.2 and 2.9 and some calculations that $b \geq 2+[(v-k)(b-r-1)^2/\{b-v-r+k+(b-2r+\lambda)(v-k-1)\}] \geq \max\{v+t-1, (\beta^2\lambda+\beta)/\alpha^2\}$. Thus, Statement 3.5 follows from Corollary 3.1, Lemma 3.2 and Theorem 2.11.

Remark 3.6. When $\alpha=1$, Corollary 3.4 has the same expression as Corollary 3.1. However, the explanation of Statement 3.3 corresponding to Corollary 3.1 is not included in that corresponding to Corollary 3.4 when $\alpha=1$. Note that when $v=2k$ (or $\beta=2\alpha$) the inequalities of Corollaries 3.1 and 3.4 become the same $b \geq 2+[k(r-1)^2/\{r-k+\lambda(k-1)\}]$.

Finally, as another case we can obtain from Theorems 2.7 and 2.11 the following

Theorem 3.7. *For an α-resolvable* BIBD *with parameters v, $b=\beta t$, $r=\alpha t$, k and λ, the following inequalities hold:*

(i) *If $v<2k$, then*

$$b \geq \max\left\{2+\left[\frac{k(r-1)^2}{r-k+\lambda(k-1)}\right], 2r-\lambda,\ v+t-1, \frac{\beta^2\lambda+\beta}{\alpha^2}\right\}.$$

(ii) *If $v>2k$ and β is not divisible by α, then*

$$b \geq \max\left\{2+\left[\frac{(v-k)(b-r-1)^2}{b-v-r+k+(b-2r+\lambda)(v-k-1)}\right], (m+1)r-m\lambda,\ v+t-1, \frac{\beta^2\lambda+\beta}{\alpha^2}\right\},$$

where $v=mk+l$ and $l\geq 1$.

We may have a further improvement for the inequalities of Theorem 3.7. Practically, it is conjectured that in (i) of Theorem 3.7

$$b\geq 2+[k(r-1)^2/\{r-k+\lambda(k-1)\}]\geq \max\{v+t-1, (\beta^2\lambda+\beta)/\alpha^2\},$$

and in (ii)

$$b\geq 2+[(v-k)(b-r-1)^2/\{b-v-r+k+(b-2r+\lambda)(v-k-1)\}]$$
$$\geq \max\{v+t-1, (\beta^2\lambda+\beta)/\alpha^2\}.$$

4. Discussion

As mentioned in Section 2, the inequalities in Theorems 2.7 and 2.11 appears to be the best for any non-symmetrical BIBD. Of course, for a BIBD with additional restrictions the inequalities may be improved further. For example, Stanton and Sprott [15] showed that if a BIBD contains $c>0$ blocks other than B_1 which are identical with a specified block B_1, then an inequality $b\geq (c+1)v-(c-1)$ holds. Mann [9] further showed that if s blocks of a BIBD with parameters v, b, r, k, λ are identical and if $r>\lambda$, then $r/k=b/v\geq s$ holds. In this case, from $c=s-1>0$ it is clear that when $s=2$, these inequalities are the very same ones, and when $s\geq 3$, Mann's inequality is more stringent than Stanton and Sprott's one. Then compare the inequalities in Theorems 2.7 and 2.11 with Mann's inequality. We first take the following examples.

Example 4.1. Consider a BIBD with $v=7$, $b=21$, $r=9$, $k=3$, $\lambda=3$ formed by taking three copies of a symmetrical BIBD with $v=b=7$, $r=k=3$ and $\lambda=1$ (see [16]). Then $b\geq sv$, $b\geq (m+1)r-m\lambda$ and

$$b\geq 2+[(v-k)(b-r-1)^2/\{b-v-r+k+(b-2r+\lambda)(v-k-1)\}]$$

become $21\geq 21$, $21\geq 21$ and $21\geq 20$, respectively.

Example 4.2. Consider a BIBD with $v=9$, $b=36$, $r=16$, $k=4$, $\lambda=6$ formed by taking two copies of a BIBD with $v=9$, $b=18$, $r=8$, $k=4$ and $\lambda=3$ (see [16]). Then $b\geq sv$, $b\geq (m+1)r-m\lambda$ and

$$b\geq 2+[(v-k)(b-r-1)^2/\{b-v-r+k+(b-2r+\lambda)(v-k-1)\}]$$

become $36\geq 18$, $36\geq 36$ and $36\geq 34$, respectively.

Example 4.3. Consider a BIBD with $v=16$, $b=32$, $r=12$, $k=6$, $\lambda=4$ formed by taking two copies of a symmetrical BIBD with $v=b=16$, $r=k=6$ and $\lambda=2$ (see [16]). Then $b \geqslant sv$, $b \geqslant (m+1)r-m\lambda$ and

$$b \geqslant 2+[(v-k)(b-r-1)^2/\{b-v-r+k+(b-2r+\lambda)(v-k-1)\}]$$

become $32 \geqslant 32$, $32 \geqslant 28$ and $32 \geqslant 32$, respectively.

As noted in Section 2, a bound of an inequality

$$b \geqslant 2+[(v-k)(b-r-1)^2/\{b-v-r+k+(b-2r+\lambda)(v-k-1)\}]$$

is attained in almost all non-symmetrical BIBDs for $v \geqslant 2k$. Thus, from Examples 4.1, 4.2 and 4.3 we have

Theorem 4.4. *For a non-symmetrical* BIBD *with parameters* v, b, r, k, λ *and with* s *identical blocks, if* $v \geqslant 2k$, *then*

$$b \geqslant \max\left\{2+\left[\frac{(v-k)(b-r-1)^2}{b-v-r+k+(b-2r+\lambda)(v-k-1)}\right], (m+1)r-m\lambda, sv\right\}$$

holds.

Similarly, we can establish the following

Theorem 4.5. *For a non-symmetrical* BIBD *with parameters* v, b, r, k, λ *and with* s *identical blocks, if* $v<2k$, *then*

$$b \geqslant \max\left\{2+\left[\frac{k(r-1)^2}{r-k+\lambda(k-1)}\right], 2r-\lambda, sv\right\}$$

holds.

Note that when $v<2k$, we have failed to construct a BIBD with s identical blocks such that $b \geqslant 2r-\lambda > sv$ or $2+[k(r-1)^2/\{r-k+\lambda(k-1)\}]$.

Finally, from the above discussions we have a feeling that inequalities (3) and (4) are the very stringent ones among non-symmetrical BIBDs with $v<2k$ and with $v \geqslant 2k$, respectively.

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